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Location of the Zeros of Polynomials Satisfying Three-Term Recurrence Relations. I. General Case with Complex Coefficients

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The finite sequences of polynomials $\{P_n\}_{n=0}^{N}$ generated from three-term recurrence relations with complex coefficients are considered. First a general method is presented which allows the determination the regions where all zeros of the polynomials in question are located. Next one way is followed, say $|\mu_n| < |\beta_n|$, and the first results are established. In the second paper (J. Gilewicz and E. Leopold, Location of zeros of polynomials satisfying three-term recurrence relations. II. General case with complex coefficients, in preparation) the reverse way, $|\mu_n| > |\beta_n|$, is followed. Subsequent papers (E. Leopold, *J. Approx. Theory* **43**, 15–24 (1985); E. Leopold, Location of zeros of polynomials satisfying three-term recurrence relations. IV. Application to some polynomials and to generalized Bessel polynomials, in preparation) are devoted to some particular cases and to numerical applications. © 1985 Academic Press, Inc.

1. INTRODUCTION

In recent papers Saff and Varga [3] and De Bruin, Saff, and Varga [4] consider the problem of location of zeros of polynomials generated from a three-term recurrence relation with positive coefficients. These relations imply the continued fraction expansion and we also start from this classical idea already applied by Sherman [1] and used again by Wall [2]. When the coefficients of the recurrence relations are known, the problem is, of course, deterministic. But the important point is to find the optimal estimations using only some general properties of these coefficients. In the present paper we use the information given by the absolute values of coefficients. Secondly we follow only one possible way of proof, where the absolute value of the ratio $\mu_n = P_{n-1}/P_n$ is always less than that of the pole of the corresponding homographic transformation: see (2.9). The second paper [7] is devoted to

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the reverse case. In the third paper [8] we apply the previous results to the relation generating the denominators of the rows in the Padé table (or numerators of columns) [5]. In particular the case of positive coefficients leads, by means of Padé approximants, to the functions of class S [5, 6]. In the fourth paper [9] we examine the relation generating the numerators or denominators of diagonals in the Padé table. Here the case of positive coefficients corresponds to the classical orthogonal polynomials and, by means of Padé approximants, to the Stieltjes functions. Many numerical results, concluding our work, are given in that paper.

In the present paper the zeros of the polynomials in question are located in annular regions. With the additional conditions on the coefficients of the recurrence relation these regions can be improved. For instance, positivity [8] allows improvement of the annular region to some moon-like region. In particular the improvement of the results of [3] and [4] will be explained in the two subsequent papers [8, 9].

2. General Case

Let $\{P_n\}_{n\geqslant 0}$ be a sequence of polynomials of respective degrees n which satisfy the three-term recurrence relation, where $B_{(n)}$ and $A_{(n)}$ are given simple polynomials:

$$\begin{gathered}\forall n \geqslant 0: P_{n+1} = B_{(n)} P_n - A_{(n)} P_{n-1}, \\ P_{-1} \equiv 0, P_0 \neq 0, \forall n \geqslant 0: \deg B_{(n)} = 1, \forall n \geqslant 1: \deg A_{(n)} \leqslant 2.\end{gathered} \tag{2.1}$$

Of course we can distinguish another natural case:

$$\forall n \geqslant 1: \deg B_{(n)} = 0, \deg A_{(n)} = 2$$

or the general recurrence relation with no real interest:

$$\forall n \geqslant 1: Q_{(n)} P_{n+1} = B_{(n)} P_n - A_{(n)} P_{n-1},$$

where the polynomials $Q_{(n)}$, $B_{(n)}$ and $A_{(n)}$ must be chosen so that all the polynomials P_k are of degree k. Though we are not concerned with these cases in this paper, the proposed method can be easily applied to them.

If in (2.1), for some fixed n and complex z, the following conditions are satisfied:

$$A_{(n)}(z)\, P_n(z)\, P_{n+1}(z) \neq 0, \tag{2.2}$$

then relation (2.1) can be replaced by

$$\frac{P_n}{P_{n+1}} = \frac{A_{(n)}^{-1}}{A_{(n)}^{-1} B_{(n)} - P_{n-1}/P_n}, \tag{2.3}$$

which can be written as follows:

$$\mu_{n+1} = \frac{\alpha_n}{\beta_n - \mu_n}, \tag{2.4}$$

where

$$\mu_n = P_{n-1}/P_n, \alpha_n = A_{(n)}^{-1}, \beta_n = A_{(n)}^{-1} B_{(n)}.$$

The last relation can be considered as one value of the following homographic transformation:

$$w_n \mapsto w_{n+1}: w_{n+1}(w_n) = \frac{\alpha_n}{\beta_n - w_n}. \tag{2.5}$$

This can be iterated with suitable conditions (2.2) and, because $\mu_0 = 0$, we obtain

$$\mu_n = (w_n \circ w_{n-1} \circ \cdots \circ w_1)(0),$$

where $f \circ g$ denotes the composition $f(g)$; this is the finite continued fraction representation of μ_n.

Let $\mathscr{A}$ denote the complex z-plane with the zeros of $A_{(n)}$ deleted; then for all z belonging to $\mathscr{A}$ we have $P_{n+1}(z) \neq 0$ if and only if $\mu_n(z) \neq \beta_n(z)$. This property follows clearly from assumptions (2.2) leading to relation (2.3). If this relation holds for all n considered, then we have the following proposition:

PROPOSITION 2.1. *Let* $\{P_n\}_{n=0}^N$ *be a finite sequence of polynomials satisfying recurrence relation* (2.1) *and*

$$\mathscr{A}_N = \{z \mid A_{(n)}(z) \neq 0 \; \forall n < N\}. \tag{2.6}$$

Then

$$\forall z \in \mathscr{A}_N, \forall n \leqslant N: P_n(z) \neq 0$$

if and only if

$$\forall n < N: \mu_n(z) \neq \beta_n(z). \tag{2.7}$$

In many particular cases the set $\mathscr{A}_N$ can be modified by addition of certain zeros of $A_{(n)}$'s which are obviously not zeros of P_n. For instance, if the only zero of $A_{(n)}$ is the point $z = 0$, then $\mathscr{A}_N$ can be replaced by $\mathbb{C}$.

Our method is aimed to exploit the relation (2.7) in order to determine the zero-free regions for polynomials. We shall use only some global information on the poles β_n: moduli, real parts or other quantities, depending essentially on the general properties of the $B_{(n)}$'s and $A_{(n)}$'s. In the present paper we consider the case of moduli, and consequently, the transformations of discs or complements of discs (with respect to discs).

The closed disc $D(0, \gamma_n)$ of radius $\gamma_n < |\beta_n|$ centered in the origin of the w_n-plane is mapped by (2.5) to the closed disc of radius ρ_{n+1} centered in ω_{n+1} in the w_{n+1}-plane:

$$\omega_{n+1} = \frac{\alpha_n \bar{\beta}_n}{|\beta_n|^2 - \gamma_n^2}, \qquad \rho_{n+1} = \frac{|\alpha_n| \gamma_n}{|\beta_n|^2 - \gamma_n^2}.$$

The last disc is imbedded in the closed disc $D(0, R_{n+1})$, where

$$R_{n+1} = |\omega_{n+1}| + \rho_{n+1} = \frac{|\alpha_n|}{|\beta_n| - \gamma_n}. \tag{2.8}$$

Except for the first step, where $\mu_0 = 0$, we can also examine the transformations of the complements of discs $D(0, \gamma_n)$ with $\gamma_n > |\beta_n|$. Finally we exploit relation (2.7) in each step, following two complementary ways:

$$|\mu_n| < |\beta_n| \tag{2.9}$$

or

$$|\mu_n| > |\beta_n|. \tag{2.10}$$

This forms a descending tree of possible ways of analysis. The mixed ways passing through (2.9) and (2.10) imply too many additional conditions for the coefficients of the recurrence relation. Since we will use particular conditions as little as possible, we analyze only two ways: condition (2.9) for all n in the present paper, and condition (2.10) for all n in the next paper [7].

If we can find bounds, say R_n, R_n^*, γ_n, γ_n^* such that

$$|\mu_n| \leqslant R_n \leqslant R_n^*, \tag{2.11}$$

$$\gamma_n^* \leqslant \gamma_n < |\beta_n|, \tag{2.12}$$

then inequality (2.9) will be always satisfied if the following inequality holds:

$$R_n^*(z) \leqslant \gamma_n^*(z). \tag{2.13}$$

Next we deduce the zero-free regions from the last inequality.

To obtain (2.11) we must examine the recursive determination of μ_n; if μ_{n-1} belongs to the disc $D(0, \gamma_{n-1})$ with $\gamma_{n-1} < |\beta_{n-1}|$, then R_n is given by (2.8). The radius γ_n can be found directly from $|\beta_n|$. The important point is to determine the optimal bounds for R_n and γ_n.

3. First Theorem; Case $|\mu_n| < |\beta_n|$

As stated, the reverse case will be examined in the next paper [7]. The conditions on the coefficients will be more complicated than in the present case.

THEOREM 3.1. *Let* $\{P_n\}_{n=0}^{N}$ *be a finite sequence of polynomials of respective degrees n which satisfy the three-term recurrence relation with complex coefficients*:

$$\forall n \geqslant 0: P_{n+1}(z) = (b_n + b_n' z) P_n(z) - (a_n + a_n' z + a_n'' z^2) P_{n-1}(z), \quad (3.1)$$

where

$$P_{-1} \equiv 0, P_0 \not\equiv 0, \forall n: b_n' \neq 0,$$

and let $\mathscr{A}_N$ *denote the following set*:

$$\mathscr{A}_N = \{z \in \mathbb{C} \mid a_n + a_n' z + a_n'' z^2 \neq 0 \ \forall n < N\}. \quad (3.2)$$

Then the region $\mathscr{P}_N$ *contains no zero of* $\{P_n\}_{n=0}^{N>1}$:

$$\mathscr{A}_N \supset \mathscr{P}_N = \mathscr{P}_N' \cup \mathscr{P}_N'',$$

where $\mathscr{P}_N'$ *and* $\mathscr{P}_N''$ *are defined in* (i) *and* (ii), *respectively.*

(i) *The following intervals,*

$$I_0 =]0, |b_0/b_0'|];$$

$$1 \leqslant m < N: I_n = \left]\left|\frac{b_n}{2b_n'}\right| - \left(\left|\frac{b_n}{2b_n'}\right|^2 - \left|\frac{a_n}{b_{n-1}' b_n'}\right|\right)^{1/2},\right.$$
$$\left.\left|\frac{b_n}{2b_n'}\right| + \left(\left|\frac{b_n}{2b_n'}\right|^2 - \left|\frac{a_n}{b_{n-1}' b_n'}\right|\right)^{1/2}\right[, \quad (3.3)$$

are nonempty if and only if the following conditions hold:

$$b_0 \neq 0; 1 \leqslant n < N: 4\,|a_n b_n'| < |b_{n-1}' b_n^2|. \quad (3.4)$$

If the intersection $\mathscr{I}_N$ *of these intervals is not empty*:

$$\mathscr{I}_N = \bigcap_{0 \leqslant n < N} I_n \neq \varnothing, \quad (3.5)$$

then the region $\mathscr{P}_N'$ *is not empty and is defined as follows*:

$$\mathscr{P}_N' = \{z \in \mathscr{A}_N \mid |z| \leqslant \max_{d \in \mathscr{I}_N} \min_{1 \leqslant n < N} x_n(d)\}, \quad (3.6)$$

where the functions x_n are defined in $\mathcal{J}_N$ by:

$$a''_n \neq 0: x_n(d) = \frac{1}{2\,|a''_n|}\,(|b'_{n-1}b'_n|\,d - |a'_n| - |b'_{n-1}b_n| + \Delta_n^{1/2}), \qquad (3.7)$$

$$\Delta_n = |b'_{n-1}b'_n|(|b'_{n-1}b'_n| - 4\,|a''_n|)\,d^2$$
$$+ 2\,|b'_{n-1}|(2\,|a''_n b_n| - |b'_{n-1}b'_n b_n| - |a'_n b'_n|)\,d$$
$$+ (|a'_n| + |b'_{n-1}b_n|)^2 - 4\,|a_n a''_n|;$$

$$a''_n = 0: x_n(d) = d - \frac{|a_n| + |a'_n|\,d}{|b'_{n-1}|(|b_n| - |b'_n|\,d) + |a'_n|}. \qquad (3.8)$$

(ii)

$$\mathcal{J}_N = [\max_{0 \leqslant n < N} d_n, \infty[\qquad (3.9)$$

exists as an interval of positive numbers, where

$$d_0 = |b_0/b'_0|,$$

$$1 \leqslant n < N: d_n = \frac{|b'_{n-1}|(|b'_{n-1}b'_n b_n| - 2\,|a''_n b_n| + |a'_n b'_n|) + 2\Delta_n'^{1/2}}{|b'_{n-1}b_n|(|b'_{n-1}b'_n| - 4\,|a''_n|)} \qquad (3.10)$$

$$\Delta'_n = |a''_n b'_{n-1}|[|a''_n b'_{n-1}b_n^2| + |a'_n b'_n|(|a'_n| + |b'_{n-1}b_n|)$$
$$+ |a_n b'_n|(|b'_{n-1}b'_n| - 4\,|a''_n|)]$$

if and only if the following conditions hold:

$$1 \leqslant n < N: 4\,|a''_n| < |b'_{n-1}b'_n|. \qquad (3.11)$$

In that case, the region $\mathcal{P}''_N$ is not empty; it contains the point at infinity and is defined as follows:

$$\mathcal{P}''_N = \bigcup_{d \in \mathcal{J}_N} \{z \in \mathcal{A}_N \mid \max_{1 \leqslant n < N} x'_n(d) \leqslant |z| \leqslant \min_{1 \leqslant n < N} x''_n(d)\}, \qquad (3.12)$$

where the functions x'_n and x''_n are defined in $\mathcal{J}_N$ by:

$$a''_n \neq 0: x'_n(d) = \frac{1}{2\,|a''_n|}\,(|b'_{n-1}b'_n|\,d - |a'_n| - |b'_{n-1}b_n| - \Delta_n^{1/2}) \qquad (3.13)$$

$$x''_n(d) = x_n(d) \qquad (x_n \text{ defined by } (3.7)), \qquad (3.14)$$

$$a''_n = 0: x'_n(d) = x_n(d) \qquad (x_n \text{ defined by } (3.8)), \qquad (3.15)$$

$$x''_n = \infty.$$

Proof. We present a constructive proof. The zero-free region shall be defined step by step by means of condition (2.9) replaced by (2.13). For better understanding we analyze the first step, i.e., $n = 0$. Condition (2.9) reduces to $|\beta_0| > 0$, that is, to the following two conditions:

$$|z| < |b_0/b_0'| \qquad \text{or} \qquad |z| > |b_0/b_0'|.$$

To define the radius γ_0 such that $\gamma_0 < |\beta_0|$, we introduce the new parameter d as follows:

$$|z| < d \leqslant |b_0/b_0'| \tag{3.16}$$

or

$$|z| > d \geqslant |b_0/b_0'|. \tag{3.17}$$

We obtain, respectively,

$$|\beta_0| = |b_0 + b_0' z| > |b_0| - |b_0'|\, d = \gamma_0$$

or

$$|\beta_0| > |b_0'|\, d - |b_0| = \gamma_0.$$

Suppose we carry out the analysis inside the disc defined by (3.16). In the second step, i.e., $n = 1$, we also have two possibilities: $|z| < |b_1/b_1'|$ or $|z| > |b_1/b_1'|$. The first one leads to the global condition $|z| < \min(|b_0/b_0'|, |b_1/b_1'|)$, however, the second one involves the additional condition $|b_1/b_1'| < |b_0/b_0'|$. Clearly the last case involves in general too many specific conditions in each step and consequently we exclude this case from our analysis. We shall do the analysis separately, for all n, first with the conditions analogous to (3.16) and afterward with these analogous to (3.17), where in both cases the inequalities must be replaced by the strict ones for $n > 0$.

(i) *The case* $|z| < d$. Suppose, for simplicity, $a_0 + a_0' z + a_0'' z^2 \equiv 1$. Then if

$$|z| < d < \min_{0 \leqslant n < N} |b_n/b_n'| \tag{3.18}$$

(in case $\min_{0 \leqslant n < N} |b_n/b_n'| = |b_0/b_0'|$, we replace (3.18) by $|z| < d \leqslant |b_0/b_0'|$), the radii γ_n and their minorants γ_n^* can be estimated as follows:

$$0 \leqslant n < N: |\beta_n| = \left| \frac{b_n + b_n' z}{a_n + a_n' z + a_n'' z^2} \right| > \frac{|b_n| - |b_n'|\, d}{|a_n + a_n' z + a_n'' z^2|} = \gamma_n$$

$$\geqslant \frac{|b_n| - |b_n'|\, d}{|a_n| + |a_n'|\,|z| + |a_n''|\,|z|^2} = \gamma_n^*. \tag{3.19}$$

Assume now that conditions (2.13) have been required up to $n-1$, then by (2.8) we can compute R_n and we obtain:

$$n \geqslant 1: |\mu_n| \leqslant R_n = \frac{1}{|b_{n-1} + b'_{n-1} z| - |b_{n-1}| + |b'_{n-1}| d}$$
$$\leqslant \frac{1}{|b'_{n-1}|(d - |z|)} = R_n^*. \tag{3.20}$$

With the help of the last estimates, condition (2.13), after rearrangement, becomes

$$n \geqslant 1: f_n(|z|) = |a''_n| \, |z|^2 + [|a'_n| + |b'_{n-1}|(|b_n| - |b'_n| d)] \, |z|$$
$$+ [|a_n| - |b'_{n-1}|(|b_n| - |b'_n| d) \, d] \leqslant 0. \tag{3.21}$$

The function f_n has one positive zero $x_n(d)$ given by (3.7) or (3.8) if and only if its constant term is negative:

$$|b'_{n-1} b'_n| \, d^2 - |b'_{n-1} b_n| \, d + |a_n| < 0. \tag{3.22}$$

The last condition holds for d belonging to $I_n =]d'_n, d''_n[$ defined by (3.3). The interval I_n is not empty iff the quadratic function in (3.22) has two real zeros d'_n and d''_n, i.e., iff condition (3.4) holds.

Observe that

$$\forall n \geqslant 1: 0 \leqslant d'_n < d''_n \leqslant |b_n / b'_n|;$$

therefore condition (3.18) is satisfied if d belongs to I_n, except for the case $n = 0$; then we must remember that $d \leqslant |b_0 / b'_0|$ because d belongs also to I_0.

However, if $N > 1$, i.e., $n \geqslant 1$, we can easily see that $x_n(d)$ from (3.7) is less than $x_n(d)$ from (3.8): the parabola f_n ($a''_n \neq 0$) defined in (3.21) lies above the straight line f_n ($a''_n = 0$). Since $x_n(d)$ from (3.8) is less than d, then for arbitrary x_n, (3.7) or (3.8), we have

$$n \geqslant 1: x_n(d) < d. \tag{3.23}$$

We look for the largest set $\{z\}$ such that $|z| < x_n(d)$ for all n with the additional condition (3.16). Then we define the lower hull x of the functions x_n by:

$$d \in \mathscr{I}_N \neq \varnothing: d \mapsto x(d) = \min_{1 \leqslant n < N} x_n(d). \tag{3.24}$$

Note that according to (3.23) the function identity $d \mapsto d$ from (3.16) does not contribute to the definition of the function x. Now we look for the unique maximum of the function x. In particular if $|b_0 / b'_0|$ is the right extremity of

$\mathscr{I}_N$, this maximum may be $x(|b_0/b_0'|)$. It must be included because $x(|b_0/b_0'|) < |b_0/b_0'|$, which assumes in fact condition (3.16). That is the reason for taking I_0 to be closed on the right. This completes the proof of the first part of the theorem leading to the zero-free region $\mathscr{P}_N'$ defined by (3.6).

(ii) *The case* $|z| > d$. In a way similar to that of (i) we choose d such that, instead of (3.18),

$$|z| > d > \max_{0 \leqslant n < N} |b_n/b_n'| \tag{3.25}$$

(except that if $\max |b_n/b_n'| = |b_0/b_0'|$, we replace (3.25) by $|z| > d \geqslant |b_0/b_0'|$). Then we must only change, in estimates (3.19), $|b_n| - |b_n'|\, d$ into $|b_n'|\, d - |b_n|$ and, in (3.20), $d - |z|$ into $|z| - d$, which, however, does not change condition (3.21). Because the constant term in the function f_n is strictly positive, inequality (3.21) has solutions only if the coefficient of $|z|$ is strictly negative, which translates for d into the following condition:

$$d > \left| \frac{a_n'}{b_{n-1}' b_n'} \right| + \left| \frac{b_n}{b_n'} \right| = \tilde{d}_n. \tag{3.26}$$

If the last condition is required, then condition (3.25) holds except for $n = 0$.

If $a_n'' = 0$, condition (3.26) is necessary and sufficient for existence of solutions (3.15) of inequality (3.21). In this case $\tilde{d}_n$ is the same as d_n given by (3.10). In particular the condition $|z| > d$ holds because, following (3.15), we have:

$$x_n'(d) > d, \tag{3.27}$$

where the strict inequality follows from condition (3.2) assuming that a_n, a_n' and a_n'' are not simultaneously 0. Finally any z such that $|z| \geqslant x_n'(d)$ satisfies inequality (3.21) for fixed n and for d belonging to $]d_n, \infty[$. As we shall see in the end of this proof, we can take instead of $]d_n, \infty[$ the interval $[d_n, \infty[$, although condition (3.21) is not satisfied for $d = d_n$.

If $a_n'' \neq 0$, the function f_n has two positive zeros x_n' and x_n'' iff condition (3.26) holds and the discriminant defined in (3.7) is nonnegative:

$$\Delta_n \geqslant 0. \tag{3.28}$$

We shall show now that only condition (3.11) leads to the nonempty region defined by (3.12). First suppose that

$$4\,|a_n''| > |b_{n-1}' b_n|.$$

Provided that the discriminant Δ_n' (see (3.10)) of the function $d \mapsto \Delta_n(d)$ is strictly positive, the function Δ_n has two zeros, say d_n' and d_n'', and inequality

(3.28) is satisfied for d belonging to $[d'_n, d''_n]$. However, we can verify that d''_n is smaller than $\tilde{d}_n$ of (3.26), which eliminates this case. In the case

$$4\,|a''_n| = |b'_{n-1}\,b'_n|,$$

inequality (3.28) is also satisfied only for d smaller than $\tilde{d}_n$. Conversely, if condition (3.11) holds, then the discriminant Δ'_n is always strictly positive, the function Δ_n has two real zeros, say d'_n and d_n, and condition (3.28) is satisfied for $d \leqslant d'_n$ or $d \geqslant d_n$, where d_n is defined by (3.10). However, once again, we can easily verify that d'_n is smaller than $\tilde{d}_n$ and finally only the solution $d \geqslant d_n$ satisfies all conditions. Consequently any z such that $x'_n(d) \leqslant |z| \leqslant x''_n(d)$ satisfies inequality (3.21) for fixed n.

Now we show that the smallest zero x'_n of f_n is greater than d. Indeed the parabola f_n is situated higher than the straight line defined by dropping the term $|a''_n|\,|z|^2$. Then the zero x'_n of this parabola is greater than the zero x'_n defined by (3.15), which is greater than d.

Finally, for arbitrary x'_n, (3.13) or (3.15), we obtain

$$n \geqslant 1 : x'_n(d) > d. \tag{3.29}$$

We look for the largest set $\{z\}$ such that $x'_n(d) \leqslant |z| \leqslant x''_n(d)$ for all n with additional condition (3.17). Then, for d belonging to $\mathscr{T}_N$, we define the higher hull x' of the functions x'_n and the lower hull x'' of the functions x''_n:

$$d \in \mathscr{T}_N : d \mapsto x'(d) = \max_{1 \leqslant n < N} x'_n(d), \tag{3.30}$$

$$d \mapsto x''(d) = \min_{1 \leqslant n < N} x''_n(d). \tag{3.31}$$

Note that according to (3.29) the function identity $d \mapsto d$ from (3.17) does not contribute to the definition of the function x'. Now we look for d belonging to $\mathscr{T}_N$ such that

$$x'(d) \leqslant x''(d) \tag{3.32}$$

and we construct the zero-free region $\mathscr{P}''_N$ defined by (3.12) as the union of annular regions where condition (3.32) is satisfied. The region $\mathscr{P}''_N$ is not empty because it contains at least the complement of some finite disc. It is obvious if all $a''_n = 0$. The asymptotic behavior of x'_n can be obtained by (3.15):

$$a''_n = 0;\, d \to \infty : x'_n(d) \approx d.$$

Conversely, provided that $a''_n \neq 0$ and dropping in (3.13) and (3.14) all terms except the linear ones, we obtain, according to condition (3.11), the following asymptotic behaviors of x'_n and x''_n:

$a_n'' \neq 0; d \to \infty$:

$$x_n'(d) \approx \left(1 + \left|\frac{a_n''}{b_{n-1}' b_n'}\right|\right) d = \gamma_n' d; \qquad 1 < \gamma_n' < 1.25,$$

$$x_n''(d) \approx \left(\left|\frac{b_{n-1}' b_n'}{a_n''}\right| - 1 - \left|\frac{a_n''}{b_{n-1}' b_n'}\right|\right) d = \gamma_n'' d; \qquad 2.75 < \gamma_n'' < \infty.$$

Then condition (3.32) is always satisfied asymptotically, which proves that the region $\mathscr{P}_N''$ contains the point at infinity.

Analyzing the case $a_n'' = 0$ we remarked that the point $d = d_n$ can be added to the interval $]d_n, \infty[$. Indeed this does not change the final result because $x_n'(d_n) = \infty$ and we have proved that ∞ belongs to $\mathscr{P}_N''$.

We conclude that the union $\mathscr{P}_N$ of the regions $\mathscr{P}_N'$ and $\mathscr{P}_N''$ contains no zero of the polynomials $P_1, \ldots, P_N$, which completes the proof. ∎

Remark. Note that the regions $\mathscr{P}_N'$ or (and) $\mathscr{P}_N''$ may be empty if conditions (3.4) or (and) (3.11) are, respectively, not satisfied. In particular the zero-free region given by Theorem 3.1 may not contain ∞, which is a zero of no polynomial $\not\equiv 0$.

The intervals of $|z|$ determined in Theorem 3.1 are illustrated by a heavy line in Fig. 1. The subintervals of the interval $\mathscr{I}_N$ where condition (3.32) is satisfied are represented by a heavy line. The knowledge of these subintervals greatly simplify formula (3.12) for the region $\mathscr{P}_N''$. In fact, in this case we can take for each subinterval the annular region between the minimum of the lower hull x' and the maximum of the upper hull x'', as in formula (3.6) for $\mathscr{P}_N'$.

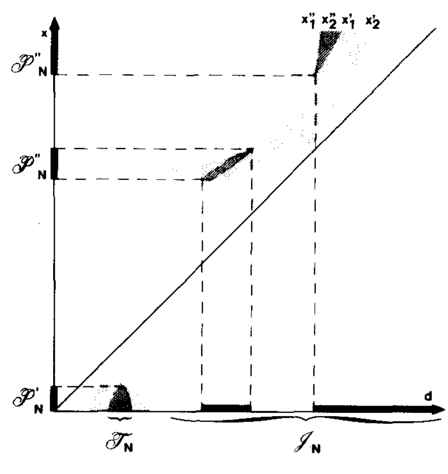

FIGURE 1

4. Application to the Padé Numerators and Denominators

The particular case of three-term recurrence relation (3.1), where $a_n = a''_n = 0$ and $b_n = 1$ [3], is the well-known Frobenius relation [6], which generates the numerators of Padé approximants along the columns, or the denominators of Padé approximants along the rows in the Padé table. We restate Theorem 3.1 in this particular situation as a corollary, because two important conditions, (3.4) and (3.11), disappear, greatly simplifying the results. This corollary, complemented by Theorem 2.1 of the third paper [8], allows one to find a moon-like region containing all zeros of the polynomials in question, as has already been mentioned in the Introduction.

Corollary 4.1. *Let $\{P_n\}_{n=0}^{N}$ be a finite sequence of polynomials which satisfy the following relation with complex coefficients*:

$$\forall n \geqslant 0: P_{n+1}(z) = (1 + t_n z) P_n(z) - u_n z P_{n-1}(z), \tag{4.1}$$

where

$$P_{-1} \equiv 0, P_0 \not\equiv 0, \forall n: t_n u_n \neq 0.$$

Then all zeros of the polynomials in question are located in the open annular region

$$\Gamma_N = \{z \in \mathbb{C} \mid \max_{d \in \mathcal{I}_N} \min_{0<n<N} x_n(d) < |z| < \min_{d \in \mathcal{I}_N} \max_{0<n<N} x_n(d)\}, \tag{4.2}$$

where

$$\mathcal{I}_N = \left]0, \min_{0<n<N} |t_n|^{-1}\left[\bigcap\right]0, |t_0|^{-1}\right], \tag{4.3}$$

$$\mathcal{I}_N = \left[\max_{0 \leqslant n<N} \frac{|t_{n-1}| + |u_n|}{|t_{n-1} t_n|}, \infty\right[\qquad (t_{-1} \equiv 1; u_0 \equiv 0), \tag{4.4}$$

$$x_n(d) = \frac{|t_{n-1}|(1 - |t_n| d) d}{|t_{n-1}|(1 - |t_n| d) + |u_n|}. \tag{4.5}$$

Sketch of the Proof. According to Theorem 3.1 in the case $a''_n = 0$, formulas (4.3), (4.4) and (4.5) follow from (3.5), (3.9) and (3.8) or (3.15), respectively. Conditions (3.4) and (3.11) being automatically satisfied, the regions $\mathcal{P}'_N$ and $\mathcal{P}''_N$ are not empty. Then the region $\mathbb{C} - \mathcal{P}'_N \cup \mathcal{P}''_N$ is a finite annular domain strictly included in $\mathbb{C}$.

5. Location of the Zeros of General Orthogonal Polynomials

The general orthogonal polynomials defined in [10, p. 40] satisfy relation (3.1), where $a'_n = a''_n = 0$. The particular case of real positive coefficients corresponds to the classical orthogonal polynomials, or, in "Padé language," to the Stieltjes case. Here condition (3.11) only disappears automatically. We restate Theorem 3.1 in this particular situation as a corollary, where the regions D_N and Γ_N are clearly the complements of the regions $\mathscr{P}''_N$ and $\mathscr{P}_N$.

Corollary 5.1. *Let $\{P_n\}_{n=0}^N$ be a finite sequence of polynomials which satisfy the following relation with complex coefficients.*

$$\forall n \geqslant 0: P_{n+1}(z) = (b_n + b'_n z) P_n(z) - a_n P_{n-1}(z), \tag{5.1}$$

where

$$P_{-1} \equiv 0, P_0 \not\equiv 0, \forall n: b'_n a_n \neq 0. \tag{5.2}$$

Then all zeros of the polynomials in question are located in the open disc

$$D_N = \{z \in \mathbb{C} \mid |z| < \min_{d \in \mathscr{T}_N} \max_{0<n<N} x_n(d)\}, \tag{5.2}$$

where

$$\mathscr{T}_N = \left[\max_{0 \leqslant n < N} \left| \frac{b_n}{b'_n} \right|, \infty \right[, \tag{5.3}$$

$$x_n(d) = d - \frac{|a_n|}{|b'_{n-1}|(|b_n| - |b'_n|\, d)}. \tag{5.4}$$

Moreover if the following conditions,

$$b_0 \neq 0; 0 < n < N: 4\,|a_n b'_n| < |b'_{n-1} b_n^2|, \tag{5.5}$$

hold, and the interval $\mathscr{T}_N$ defined by (3.5) *is not empty, then the disc D_N reduces to the open annular region*:

$$\Gamma_N = \{z \in \mathbb{C} \mid \max_{d \in \mathscr{T}_N} \min_{0<n<N} x_n(d) < |z| < \min_{d \in \mathscr{T}_N} \max_{0<n<N} x_n(d)\}. \tag{5.6}$$

Conclusions

The analysis of the general three-term recurrence relation allows one to take advantage of the subtleties of the method presented here. The general result leads automatically to many particular cases which are of greatest importance in practice. Clearly these results can be further simplified if the coefficients of the recurrence relation present some specific properties. Some of these cases are studied in subsequent papers [7–9].

REFERENCES

1. J. SHERMAN, On the numerators and the convergents of the Stieltjes continued fractions, *Trans. Amer. Math. Soc.* **35** (1933), 64–87.
2. H. S. WALL, "Analytic Theory of Continued Fractions," Van Nostrand, New York, 1948.
3. E. B. SAFF AND R. S. VARGA, Zero-free parabolic regions for sequences of polynomials, *SIAM J. Math. Anal.* **7** (1976), 344–357.
4. M. G. DE BRUIN, E. B. SAFF, AND R. S. VARGA, On the zeros of generalized Bessel polynomials, I and II, *Nederl. Akad. Wetensch. Indag. Math.* **84** (1981), 1–25.
5. J. GILEWICZ, "Approximants de Padé," Lecture Notes in Math. No. 667, Springer-Verlag, Berlin/New York/Heidelberg, 1978.
6. E. LEOPOLD, "Approximants de Padé pour les fonctions de classe S et localisation des zéros de certains polynômes," Thesis, Université d'Aix-Marseille I, 1982.
7. J. GILEWICZ AND E. LEOPOLD, Location of the zeros of polynomials satisfying three-term recurrence relations. II. General case with complex coefficients, in preparation.
8. E. LEOPOLD, Location of the zeros of polynomials satisfying three-term recurrence relations. III. Positive coefficients case, *J. Approx. Theory* **43** (1985), 15–24.
9. E. LEOPOLD, Location of the zeros of polynomials satisfying three-term recurrence relations. IV. Application to some polynomials and to generalized Bessel polynomials, in preparation.
10. E. BREZINSKI, "Padé-type Approximation and General Orthogonal Polynomials," Birkhaüser-Verlag, Basel, 1980.